GRADE 12 EXAMINATION
NOVEMBER 2011

ADVANCED PROGRAMME MATHEMATICS

Time: 3 hours 300 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 19 pages, an Answer Sheet of 2 pages and an Information Booklet of 4 pages (i – iv). Please check that your question paper is complete.

2. This question paper consists of FOUR Modules:

   MODULE 1: CALCULUS AND ALGEBRA (200 marks) is compulsory.

   Choose **ONE** of the **THREE** Optional Modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used, unless otherwise indicated.

4. All necessary calculations must be clearly shown and writing should be legible.

5. Diagrams have not been drawn to scale.

6. If applicable, calculations should be done using radians and answers should be given in radians.

7. Answer each module in a separate Answer Book.

8. Round off your answers to two decimal digits, unless otherwise indicated. For the Statistics Module, give probabilities correct to 4 decimal places.

MODULE 1 CALCULUS AND ALGEBRA

QUESTION 1

1.1 The points $P(3;p)$ and $Q(6;q)$ lie on the curve: $y = 3ln\,x$.
Determine the gradient of the line PQ and give the answer in log form. (7)

1.2 $f(n) = 15^n - 8^{n-2}$. Determine the value of p if $f(n+1) - 8f(n) = p \times 15^n$. (8)

[15]

QUESTION 2

2.1 Jerry wants to prove the formula that is sometimes used in a Riemann sum:

$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, thus that, $1 + 8 + 27 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$ for $n \in N$.

He does the following steps:

Step	
1	Let $n = 1$:
2	LHS = 1 RHS = 1
3	$\therefore$ the statement is true for $n = 1$.
4	Let $n = k$:
5	$1 + 8 + 27 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$
6	Let $n = k + 1$:
7	RHS = $\frac{(k+1)^2(k+2)^2}{4}$
8	LHS =
9	Conclusions

(a) Jerry made a mistake in Step 4. Write the correct version for this step. (1)

(b) Jerry could not finish Step 8. Write down this part of the proof. (8)

2.2 **Use Riemann sums** to calculate the exact value of the area bounded by the graph of $f(x) = x^3 + 1$ and the x-axis, between the lines $x = 0$ and $x = 2$. (11)

[20]

QUESTION 3

Question 3.1 must be answered on the Answer Sheet.

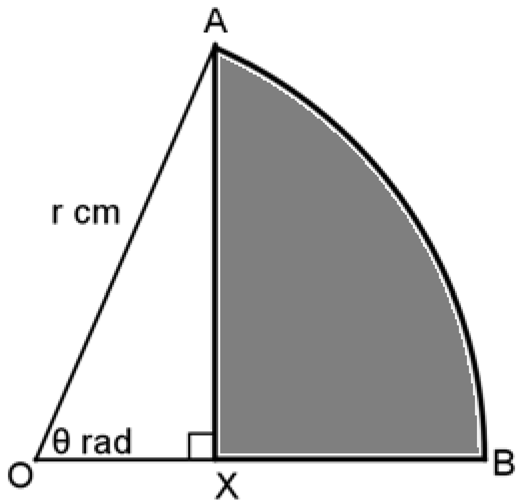
3.1 The graphs of $f(x)=\left|\frac{15}{x}\right|$ and $g(x)=|x-2|$ intersect when $x=-3$ and $x=5$.
Draw sketch graphs of $f(x)$ and $g(x)$ on the set of axes drawn on the Answer Sheet. (7)

3.2 Hence, or otherwise, solve for x if : $\left|\frac{15}{x}\right| \geq |x-2|$. (4)

[11]

QUESTION 4

In the diagram, AB is an arc of a circle, centre O and radius r cm, and $A\hat{O}B=\theta$. The point X lies on OB and AX $\perp$ OB.



4.1 Show that the area, A, of the shaded region AXB is given by
$A=\frac{1}{2}r^2(\theta-\sin\theta\cos\theta)$ cm^2. (7)

4.2 In the case where $r=12$ and $\theta=\frac{1}{6}\pi$, find the perimeter of the shaded region AXB, leaving your answer in terms of $\sqrt{3}$ and π. (8)

[15]

QUESTION 5

The polynomial $P(x) = x^5 + 5x^4 - 2x^3 - 25x^2 - 29x - 22$ is given. Two of the zeros of this polynomial are $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $-1+2\sqrt{3}$.

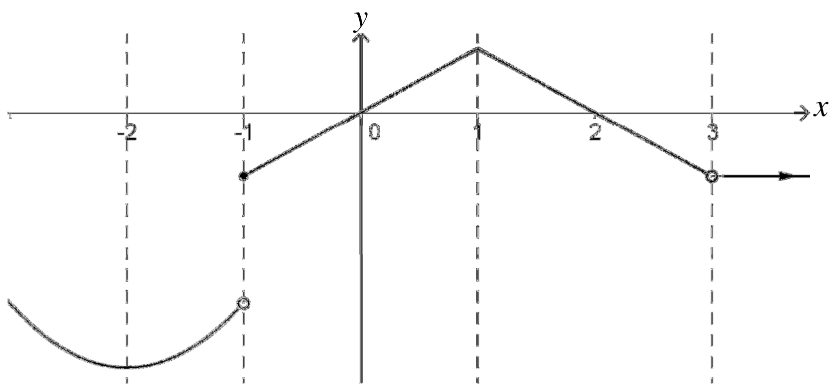
5.1 Use this information to write $P(x)$ as the product of two quadratic factors and one linear factor. (14)

5.2 Write down the real root(s) of the equation $P(x) = 0$. (2)

[16]

QUESTION 6

6.1 The function of f is represented by the graph below:



(No reasons are required in these questions)

Give in each of the following questions all the values of x for which

(a) the limit exists, but the function is not defined. (2)

(b) the left- and right-hand limits both exist, but they are unequal. (2)

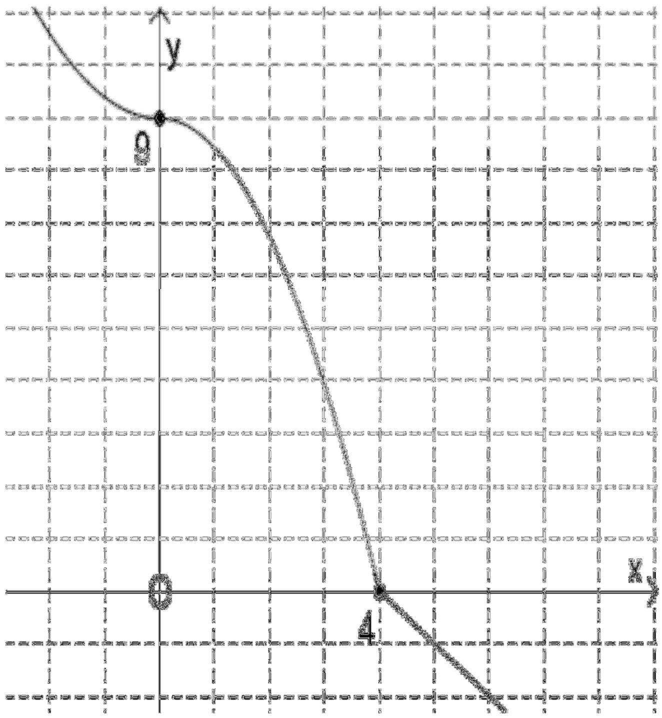
(c) f is continuous but not differentiable. (2)

(d) $f'(x) = 0$. (3)

(e) $f''(x) > 0$. (3)

6.2 The diagram shows the graph of the function

$$f(x) = \begin{cases} px^n + q \text{ if } x < 4;\ n \in N \\ -x + 4 \text{ if } x \geq 4 \end{cases}$$



This question must be answered on the Answer Sheet.

Using the diagram and **without solving for *p* and *q***, draw separate sketches on the sets of axes provided on the Answer Sheet of the following, clearly showing all intercepts with the axes, and the asymptotes where applicable:

(a) $y = |f(x)|$ (3)

(b) The inverse function, $y = f^{-1}(x)$. (5)

[20]

QUESTION 7

7.1 Determine $\dfrac{dy}{dx}$ if $y = \sqrt[3]{x}.\cos(x^2 - 1)$. (9)

7.2 (a) Find $f'(x)$ and $f''(x)$ if $f(x) = \cot x$. (4)

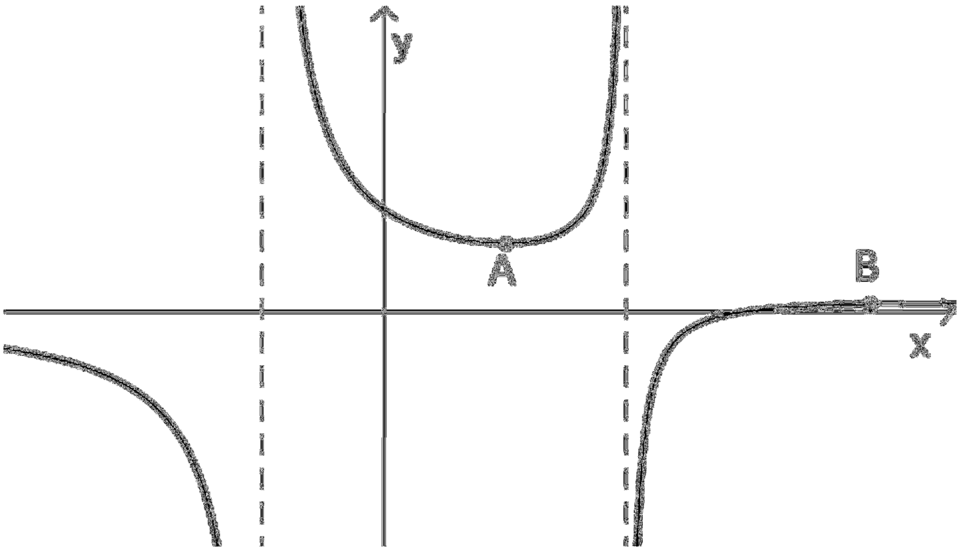
(b) This function has a non-stationary point of inflection if $f'(x) \neq 0$ and $f''(x) = 0$. Show that f has a non-stationary point of inflection at $x = \dfrac{\pi}{2}$. (5)

7.3 Determine the equation of the tangent to the curve $2x + 3y - xy^2 + 4 = 0$ at the point (–1; –2). (13)

[31]

QUESTION 8

8.1 The sketch shows the graph with equation $f(x) = \dfrac{3x-9}{x^2-x-2}$. A is the relative minimum turning point on the curve.



(a) **Give** the equations of all the vertical and horizontal asymptotes of the graph of *f*. (5)

(b) Calculate the $x-$co-ordinate of A. (10)

(c) Explain why there is only one valid answer for Question 8.1 (b). (1)

8.2 The functions $f(x) = \tan\,(x - \dfrac{\pi}{4})$ and $g(x) = \dfrac{x}{2}$ intersect at a point close to $x = \dfrac{\pi}{2}$. Use **one** application of Newton's method to find the next approximation to this solution. Give the answer correct to four decimal digits. Show how you used Newton's method. (6)

8.3 The number of visitors to a tourist attraction for a one year period can be calculated using the equation $C(t) = t\sqrt{1-t}$, where *t* is a fraction of a year and *C* is the number of visitors, in hundreds. Calculate *t* when there was a maximum number of visitors. Give the answer as a fraction. (9)

[31]

QUESTION 9

9.1 Determine the following integrals:

(a) $\int \frac{2}{\sqrt{5x-1}}\,dx.$ (5)

(b) $\int \frac{2x}{\sqrt{5x^2-1}}\,dx.$ (6)

(c) $\int \frac{2x}{\sqrt{5x-1}}\,dx.$ (Hint: Use integration by parts) (9)

9.2 (a) Express $\frac{1}{x^3+x^2}$ in partial fractions, in the form $\frac{a}{x}+\frac{b}{x^2}+\frac{c}{x+1}$. (6)

(b) Given that $\int \frac{1}{mx+c}\,dx = \frac{1}{n} ln\left|mx+c\right|$, use your answer from Question 9.2 (a) and determine $\int \frac{1}{x^3+x^2}\,dx$, leaving your answer in simplest log form. Use $a = -1$, $b = 1$ and $c = 1$. (5)

[31]

QUESTION 10

10.1 Prove that $(\sin x + \cos x)^2 = 1 + \sin 2x.$ (2)

10.2 The volume generated when the graph of $y = sinx + cosx$ rotates around the x-axis between $x = 0$ and $x = a$ is equal to $\pi(a+1)$. Use Question 10.1, or otherwise, to calculate the smallest positive value of a. (8)

[10]

Total for Module 1: 200 marks

MODULE 2 STATISTICS

QUESTION 1

In *Hugo's Café* 70% of customers buy a cup of tea.

In a random sample of 10 customers find the probability that exactly 8 buy a cup of tea.

[9]

QUESTION 2

A certain medical condition occurs in 2% of the population. The Mac Gregor blood test is used to diagnose the condition. In 9 out of 10 cases where the patient has the disease, it produces a positive result. If the patient does not have the disease there is still a 0,06 chance that the test will give a positive result. Jessie has taken the test and her result is positive. Using a tree diagram, or otherwise, find the probability that she has the disease.

[10]

QUESTION 3

Street vendors in the centre of Johannesburg sell cups of hot chocolate and packets of cooked chips during lunch hour every day. They kept records of the number of cups of hot chocolate and the number of packets of chips sold and the temperature at lunchtime on eight days.

Temperature (°C) x	12	13	14	16	18	22	24	26
Cups of chocolate y_1	300	280	250	210	190	170	150	130
Packets of chips y_2	280	160	250	120	294	206	238	240

3.1 The correlation coefficient of temperature and cups of hot chocolate sold is –0,966. Give your interpretation of this correlation coefficient in realistic terms. (3)

3.2 The correlation coefficient of temperature and packets of chips sold is 0,104. Give your interpretation of this correlation coefficient in realistic terms. (2)

3.3 The least squares regression lines for the sales of hot chocolate and packets of chips are, respectively:

$y_1 = 413 - 11{,}22x$ and $y_2 = 202 + 1{,}16x.$

Use this information to predict an estimate for the following:

(a) the number of cups of hot chocolate that will be sold when the temperature is 9 °C. (2)

(b) the number of packets of chips that will be sold when the temperature is 21 °C. (2)

3.4 Comment on the reliability of the predictions in Question 3.3 and also the limitations of these models. (4)

[13]

QUESTION 4


[<www.dailymail.co.uk>]

The final event at the Olympic Games is always the marathon.

From statistics gathered in the Beijing Olympics, the average time run by the competitors was 2 hours 14 minutes with a standard deviation of 16 minutes. The organisers divide the athletes into three equal categories (fast, medium and slow).

If the times are normally distributed, calculate the range of times for which a runner can be in the medium category.

[10]

QUESTION 5

A standard pack of playing cards has 52 cards (excluding 'jokers'). There are four suits – clubs, diamonds, hearts and spades. The value of the cards in each suit range from two to ten and then each suit has a Jack, a Queen, a King and an Ace.

In the game of poker each player is dealt five cards from the pack.

5.1 A 'flush' means that the player is dealt five cards from the same suit. What is the probability that a randomly selected player will be dealt a flush? (8)


[<www.cardwolves.com>]

5.2 A 'full house' means that a player is dealt a pair and three of a kind, e.g. three sevens and two kings or three jacks and two eights, etc. Find the probability that a randomly selected player will be dealt a 'full house'. (8)

[16]

QUESTION 6

A random variable has a probability density function given by

$$f(x) = \begin{cases} k\,(3x^2 - x^3) & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

6.1 Show that $k = \dfrac{4}{27}$. (8)

6.2 The mode of a continuous data set is the value of the random variable where it is most dense, i.e. where the density function reaches its maximum value.
Find the mode. (8)

[16]

QUESTION 7

The *Munda* manufacturing company makes battery operated children's toys. *Munda* uses batteries made by two different companies: *Dura Power* and *Long Life*. A random sample of 100 of each type of battery was selected and tested. Information on the two different types of batteries is given in the table below.

	Mean battery life	Standard deviation
Dura Power	58,5 minutes	20 minutes
Long Life	53,5 minutes	19 minutes

7.1 Is there a significant difference between the mean life span of these two batteries at the 5% significance level? (11)

7.2 The *Munda* manufacturing company would like to reject the null hypothesis. What should the exact significance level be in order to do this? (6)

[17]

QUESTION 8

[<smh.co.au>]

South Africa will be sending 10 female swimmers to the 2012 London Olympics. They will be allocated three rooms in the Olympic village. One room has five beds, one has four beds and the other has 2 beds.

In how many different ways can the female swimmers be accommodated?

[9]

Total for Module 2: 100 marks

MODULE 3 FINANCE AND MODELLING

QUESTION 1

A prospective home owner saves a certain amount every month to put down a deposit on his first home, which he plans to purchase in 30 months' time. He secures an investment rate of 7,4% per annum, compounded monthly, from a local bank.

1.1 Calculate the value of the deposit he will have, if he plans to save R3 200 at the end of every month, starting in one month's time. He intends to make the final payment at the end of the 30 months. (6)

1.2 Assuming that a deposit of R105 000 is required for his home in 30 months' time, calculate what the prospective home owner will need to save each month if he were to make a total of 30 payments, with each payment being made at the **beginning** of each month, starting immediately. (8)

[14]

QUESTION 2

Sandra, a contestant in a television game show won R150 000. She chose to invest her winnings in an account that earns 6,25% per annum, compounded monthly, with the intention of withdrawing R3 400 at the end of every month for basic living expenses. She will start to draw the money in three months' time.

2.1 Calculate the value of Sandra's investment at the time she wants to make her first withdrawal of R3 400. (4)

2.2 Calculate the amount by which Sandra's initial investment will decrease immediately after her first withdrawal. (4)

2.3 Calculate the balance in the account at the end of the four years. (10)

2.4 Six months before the end of this four year period, Sandra had to withdraw an **additional** R10 500 to pay for a small operation, followed by another **additional** withdrawal of R5 000 two months later for further medical bills. She continues to withdraw R3 400 per month as usual. Show relevant calculations to determine whether or not she would be in overdraft at the end of these four years. (10)

[28]

QUESTION 3

Thirisan secured a bank loan of R36 000 to purchase a second-hand car. The compound interest charged on this loan is 8,2% **per quarter**. He agrees to repay R4 000 per quarter, starting in 3 months' time.

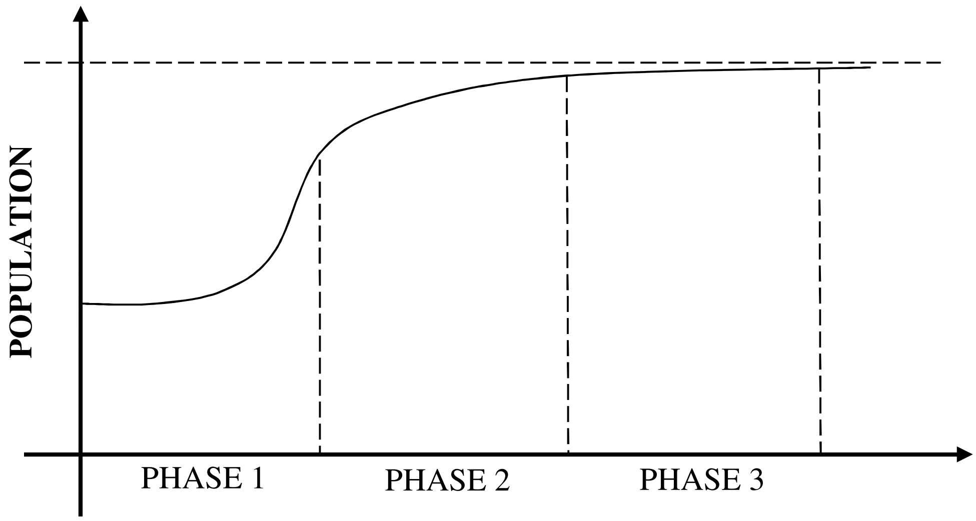
Calculate the total number of quarterly payments Thirisan will make in order to fully repay the loan.

[8]

QUESTION 4

The Karoo National Park near Beaufort West made headlines in 2010 by re-introducing lions to the Park. The last recorded lion in this area was shot in 1846. There is plenty of zebra and antelope on which the lions can feed. The lions are without natural predators.

4.1 The graph below records the projected lion population over the next century.



(a) Does this graph represent a Malthusian or Logistic population model? Give a reason for your answer. (2)

(b) During which phase is the greatest numerical growth in the lion population? Explain how this can be read off the graph. (2)

(c) Explain what is happening during the last phase, with regard to the lion population. (2)

4.2 In 2010, eight lions were re-introduced to the Karoo National Park, which can probably sustain 20 lions, with an intrinsic growth rate of 0,104 per birth cycle. A lioness has a litter once every **two** years.

(a) Set up a recursive formula to calculate the lion population at the end of twelve years, given that one lion is lost to poaching every two years. Comment on the sustainability of the lion population with this model. (8)

(b) Determine the lion population after twelve years if the Park manages to improve the poaching situation so that one lion is lost only every four years. Comment on your findings. (4)

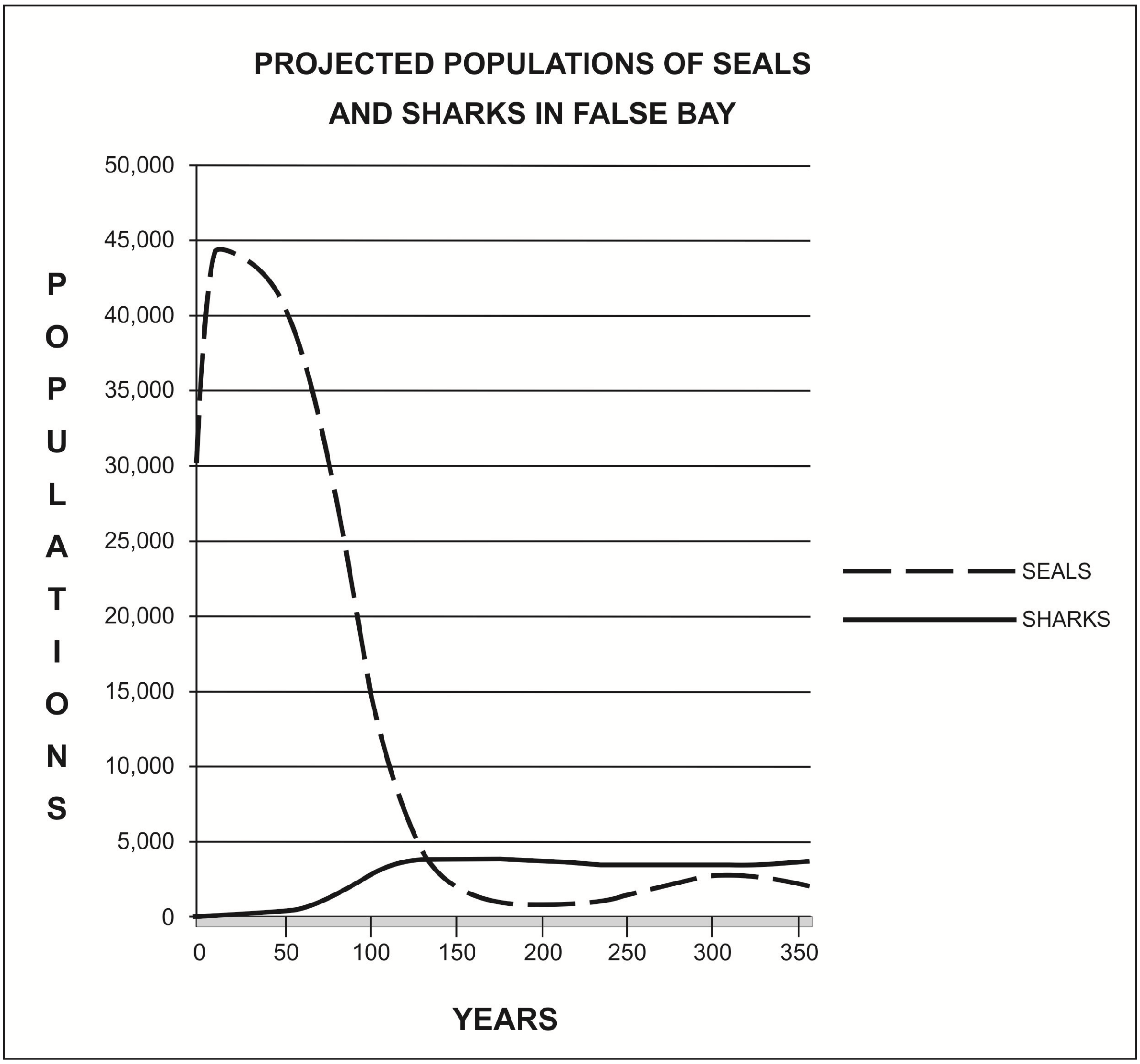
4.3 The average size of a litter is three cubs, but their survival rate is surprisingly low. Half of all cubs born are female. In the wild, lions average a life span of 12 years. Calculate the survival rate of a cub, to the nearest percent, for an intrinsic growth rate of 0,104 per birth cycle of two years. (6)

[24]

QUESTION 5

In the middle of False Bay is Seal Island, a popular tourist attraction. Seal Island is home to tens of thousands of Cape Fur Seals. False Bay is also home to Great White Sharks. The Cape Fur Seal forms about 60% of the diet of the Great White Shark.

Using the Lotka-Volterra model, nature conservationists plotted the projected populations of both groups for the next 350 years. The graph below represents their findings.



5.1 Refer to the graph to answer the following questions:

(a) Does the graph indicate that the populations tend to an equilibrium, or that the populations are cyclic? Give a reason for your answer. (2)

(b) Determine the number of years for which the seal population is expected to exceed 30 000. (2)

(c) It is proposed that 5 000 seals be culled soon, due to the fact that their projected population is to increase by more than 10 000 in the next few years. Discuss the long-term effect this could have on the seal population. (2)

(d) Consider the projected populations over the next 350 years. Comment on whether or not you think that this model is realistic. Substantiate your answer. (2)

5.2 The two populations are modelled by the recursive formulae:

$$C_{n+1} = C_n + a.C_n\left(1 - \frac{C_n}{K}\right) - b.C_n.W_n$$

$$W_{n+1} = W_n + f.b.C_n.W_n - c.W_n$$

(a) Calculate the average lifespan of a Great White Shark if the value of the parameter c is given as 0,033 333. (2)

(b) Explain the meaning of the term $b.C_n.W_n$. (2)

(c) Explain the meaning of the parameter b. (2)

(d) Calculate the value of the parameter b, correct to four decimal places, from the given data:

In terms of the populations: $C_0 = 30\ 000$, $C_1 = 33\ 000$ and $W_0 = 30$
Also for this model: $a = 0{,}38$ and $K = 44\ 000$ (6)

[20]

QUESTION 6

Consider the sequence defined by the second order recursive formula:

$T_n = p.T_{n-1} + q.T_{n-2}$

If we take the ratio of consecutive terms of the sequence, $\frac{T_n}{T_{n-1}}$, we find that, as n gets larger, the value of the ratio tends to the largest absolute solution of the equation $x^2 - px - q = 0$.

Here is an example to demonstrate this relationship:

For $T_n = -3.T_{n-1} + 2.T_{n-2}$ the ratio of consecutive terms $\frac{T_n}{T_{n-1}}$ tends to $-3{,}56155$.

The quadratic equation $x^2 + 3x - 2 = 0$ has solutions $-3{,}56155$ and $0{,}56155$.

Note that $|\ -3{,}56155\ | > |\ 0{,}56155\ |$.

The well-known Fibonacci number sequence is described by the second order recursive formula $T_n = T_{n-1} + T_{n-2}$, where $T_1 = T_2 = 1$.

6.1 Give the quadratic equation that corresponds to the second order recursive formula for the Fibonacci sequence. (2)

6.2 Hence determine for the Fibonacci sequence, correct to five decimal places, the limit of the ratio $\frac{T_n}{T_{n-1}}$ as n increases. (4)

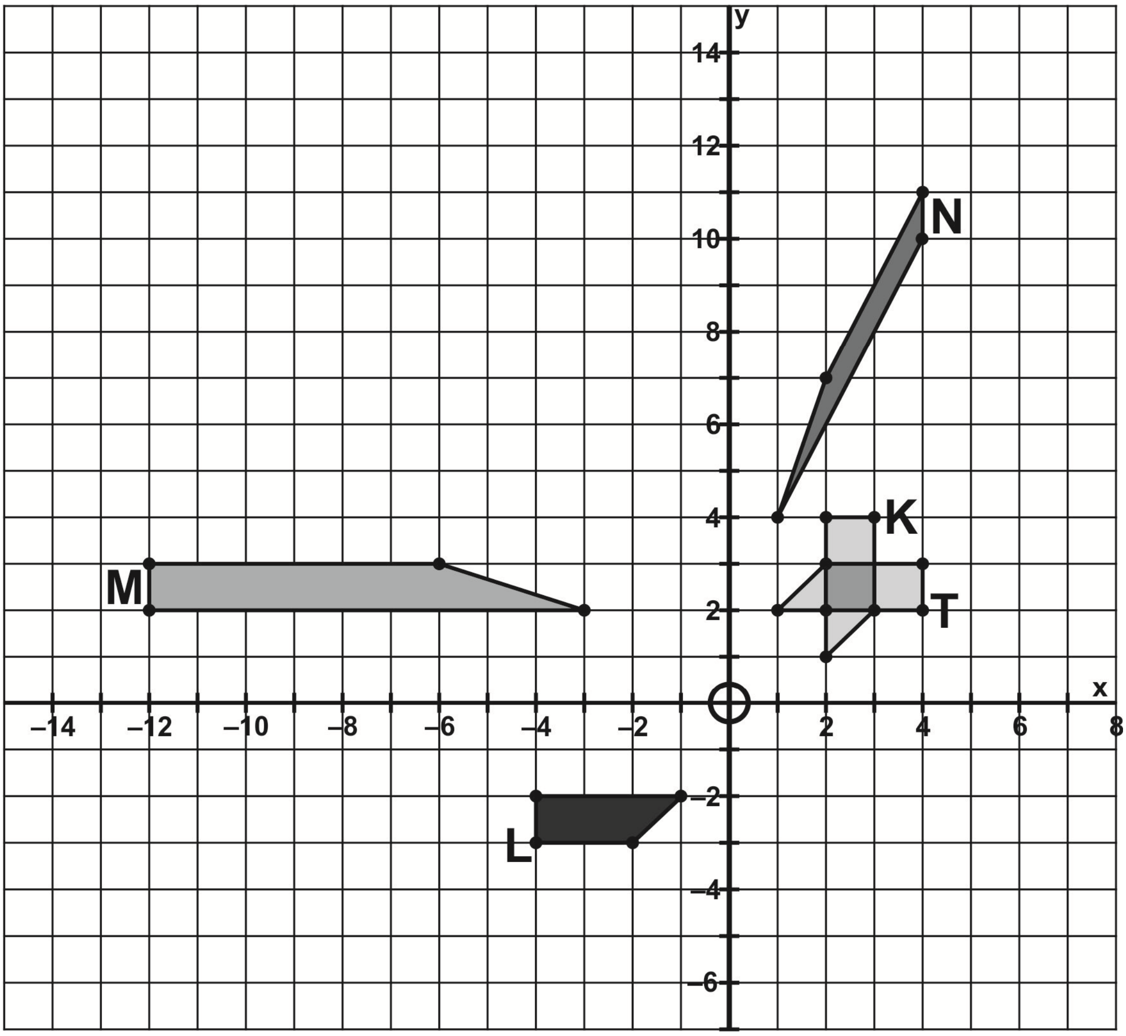
[6]

Total for Module 3: 100 marks

MODULE 4 MATRICES AND GRAPH THEORY

QUESTION 1

The trapezium T has vertices with coordinates (1;2), (2;3), (4;3) and (4;2).
Shapes K, L, M and N are images of T under different transformations. These transformations are illustrated on the diagram below.



1.1 Describe in detail the following transformations in words:

(a) $T \rightarrow K$ (2)

(b) $T \rightarrow M$ (4)

1.2 Quote the matrix that would effect the following transformation:

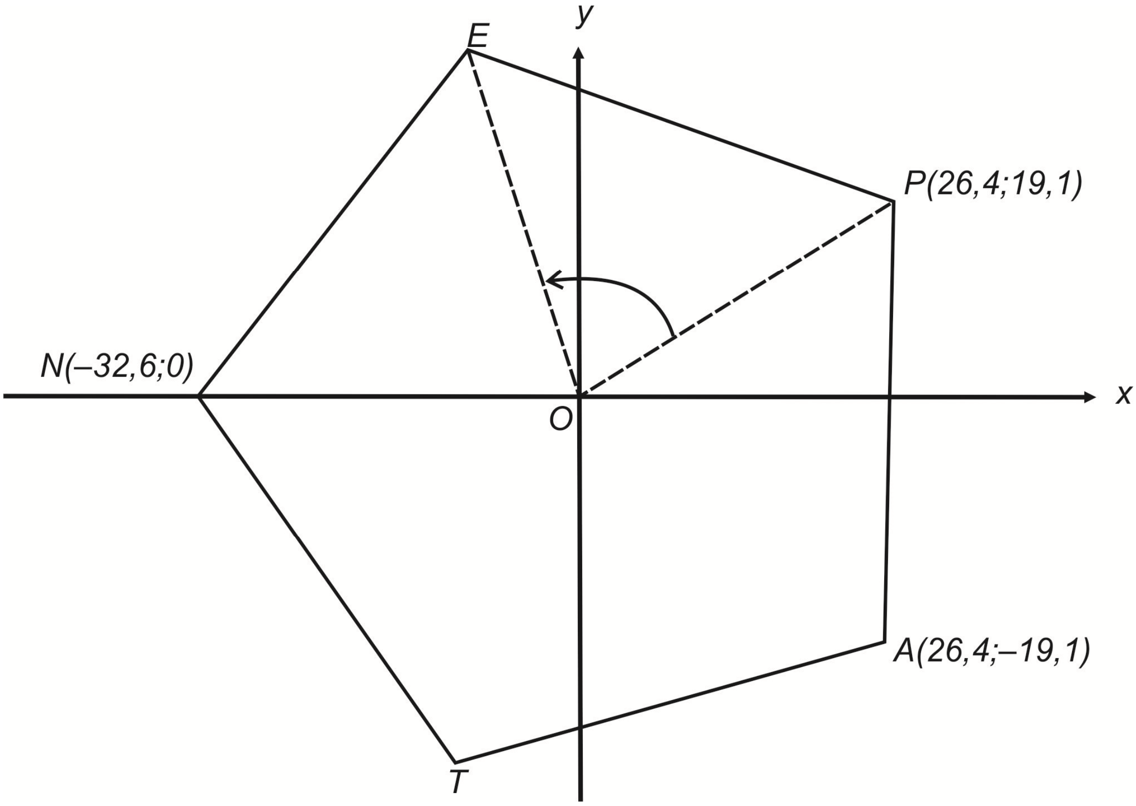
(a) $T \rightarrow L$ (2)

(b) $T \rightarrow N$ (4)

[12]

QUESTION 2

PENTA is a regular pentagon, formed by rotating point P(26,4;19,1) about the origin.
The coordinates of two other vertices are also given: A(26,4;–19,1) and N(–32,6;0).



2.1 Explain why $P\hat{O}E = 72^{o}$. (2)

2.2 Use a matrix calculation to determine the coordinates of E, correct to the nearest integer. (6)

2.3 PENTA is the image of another regular pentagon, after an enlargement through the origin. If the area of PENTA is k times the area of the original pentagon, give the matrix in terms of k, that maps the original figure onto PENTA. (2)

2.4 Line segment PA is to be reflected about a line with equation $y = mx$.
The images of these respective points are then P'(31;–10) and A'(19,2;26,3).
Find the angle of inclination of the line of reflection, correct to the nearest degree. (12)

[22]

QUESTION 3

3.1 Given matrix $P = \begin{pmatrix} -1 & 1 & 5 \\ 2 & 4 & 1 \\ -2 & 2 & 3 \end{pmatrix}$

Find the matrix Q so that $PQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (10)

3.2 Solve the following equations simultaneously, using Gaussian reduction.
Be sure to show relevant working in the process of obtaining the solutions.

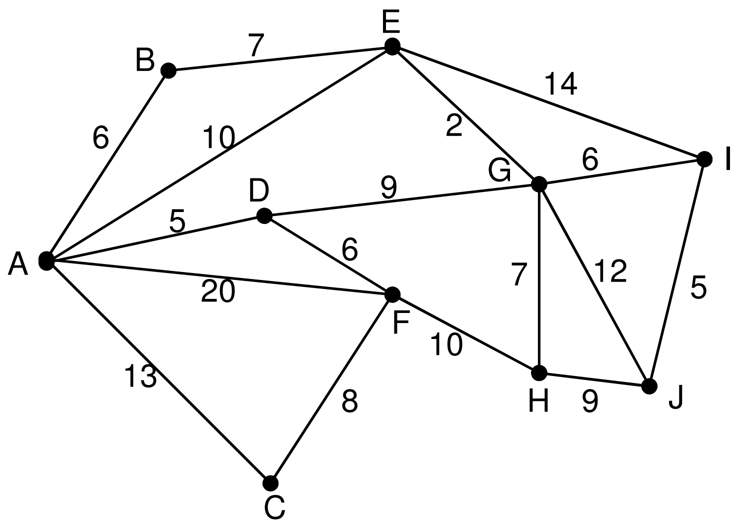
$-2x + z = 5$ $\qquad x - y - 3z = -15$ $\qquad x + y - 2z = 6$ (10)

[20]

QUESTION 4

The graph represents a circuit board, with the edges showing electrical resistance, measured in micro-ohms.

An electric current has its source at vertex A, and must reach vertex J through the path of least resistance. It does not have to pass through every vertex, nor does it necessarily have to pass along every edge.



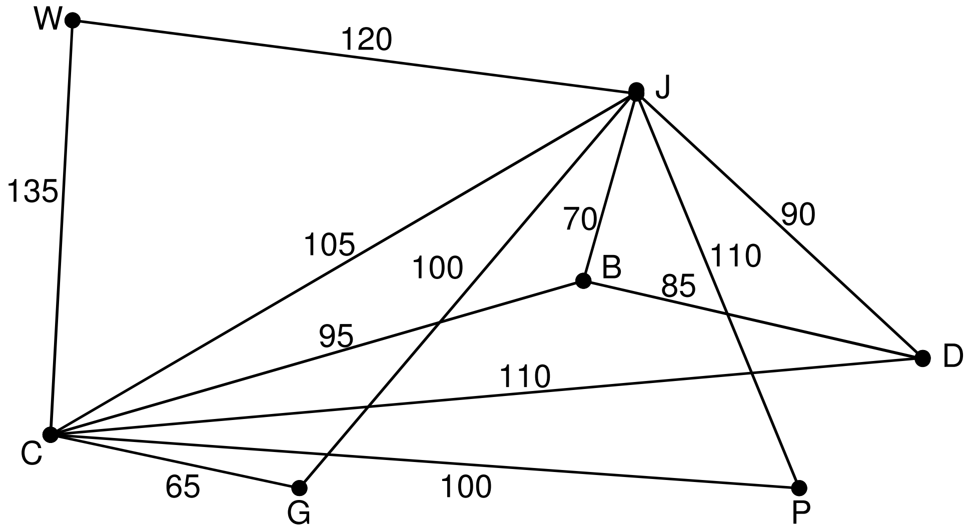
Use Dijkstra's Algorithm to find the path of least resistance from vertex A to vertex J.
Clearly state the path of least resistance, as well as the minimum resistance along this path.

[10]

QUESTION 5

An airline flies routes between major cities in southern Africa. The graph represents the cities it lands at, with the edges representing the flying time between cities in minutes.

A businessman resides in George (vertex G). He needs to visit each of the cities, shown in the diagram, once before returning home.



5.1 Find a lower bound for the time he needs to travel, starting and returning to George. Begin the process by initially removing the vertex B. (8)

5.2 Find an upper bound for his travelling time, using the Nearest Neighbour Algorithm, starting and ending in George. (8)

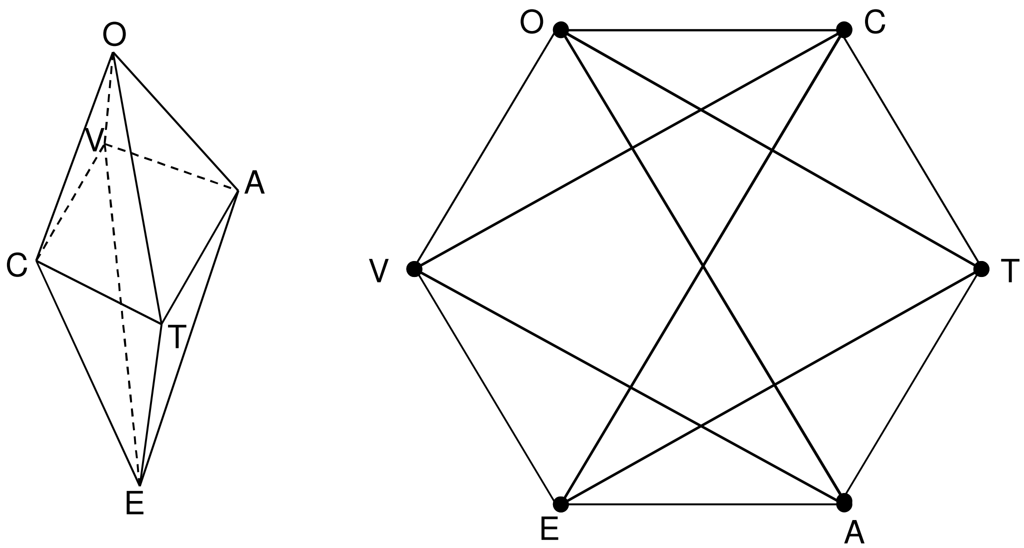
5.3 Is the answer in Question 5.2 a good estimate for an upper bound? Give a reason for your answer. (2)

5.4 By inspection, find a 'good' route for him to fly from George, visiting each city before returning home. State this route, as well as the total flying time of the route. (8)

[26]

QUESTION 6

OCTAVE is an octahedron, with its corresponding graph drawn alongside it.



6.1 Does OCTAVE represent a regular graph? Give a reason for your answer. (2)

6.2 Is it possible to find an Eulerian Circuit in OCTAVE without additional edges being created? Give a reason for your answer. (2)

6.3 A graph is said to be 'planar' if it can be drawn in a two dimensional plane without any edge crossing any other edge. Redraw the graph of OCTAVE to demonstrate that it is indeed planar. (6)

[10]

Total for Module 4: 100 marks

Total: 300 marks